Targeting Student Reasoning and Sense Making in Algebra 1

Edward C. Nolan

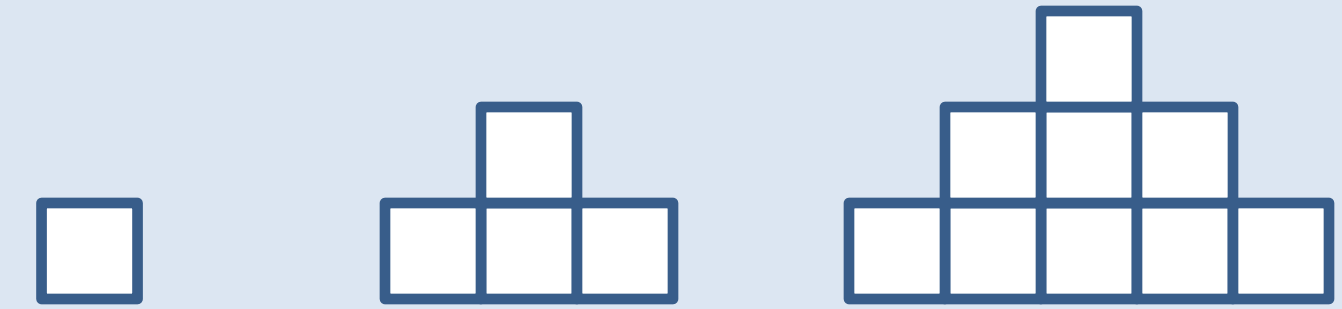
ed@nolanmath.com

Montgomery County Public Schools

Rockville, Maryland

Setting the Stage

In the figure, as the step changes, _____ also changes.



Step 1 Step 2 Step 3

Peterson, Blake E. “Linear and Quadratic Change: A Problem from Japan,” *Mathematics Teacher*, NCTM: Reston, VA, October 2006. Pages 206-212.

Setting the Stage Solutions

The following list gives some examples of student responses:

perimeter
height
width
size of enclosing rectangle
number of "toothpicks"
number of interior toothpicks
number of intersections
number of corners
number of convex corners
number of squares

number of nonadjacent squares
number of right angles
sum of the interior angles
number of diagonals
leftover space
number of segments
number of parallel lines
length of longest line
number of rectangles

Peterson, Blake E. "Linear and Quadratic Change: A Problem from Japan," *Mathematics Teacher*, NCTM: Reston, VA, October 2006. Pages 206-212.

What is Engagement?

"(S)tudents can be more successful when they are engaged in doing mathematics -

- writing about mathematics,
- modeling mathematical situations,
- discussing mathematics,
- exploring mathematical ideas

rather than watching their teacher do mathematics."

Seeley, Cathy. "Engagement as a Tool for Equity," *NCTM News Bulletin*, National Council of Teachers of Mathematics: Reston, VA, November 2004.

What is Engagement?

"The teacher should observe students in their efforts, prompt them with guiding questions, and help them generalize results."

Freeman, Gregory d. and Lisa B. Lucius. "Student Engagement and Teacher Guidance in Meaningful Mathematics: Enduring Principles," *Mathematics Teacher*, National Council of Teachers of Mathematics: Reston, VA, October 2008. Pages 164-167.

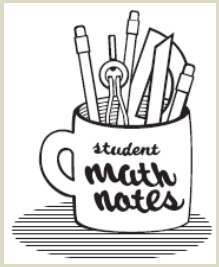
Problem solving in meaningful contexts

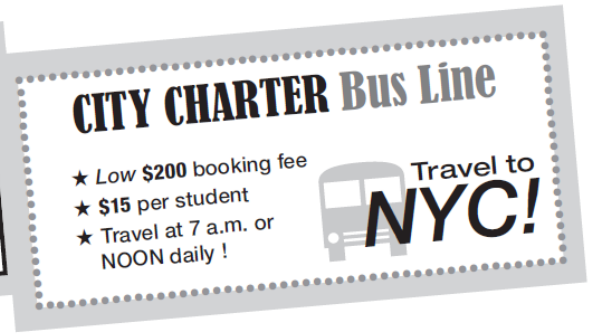
Lupe wants to earn money in the summer by mowing lawns. The going rate is $7.50 per hour if Lupe uses the customer's lawn mower. How much money can Lupe earn in a summer?

This problem can be changed slightly. Suppose Lupe considers buying her own lawn mower for $250 and charging her customers $12 per hour. Which plan will give Lupe the most profit?

Phillips, Elizabeth. Patterns and Functions, Addenda Series Grades 5-8. National Council of Teachers of Mathematics (NCTM): Reston, VA, 1991.

SMN's *Big Apple or Bust*





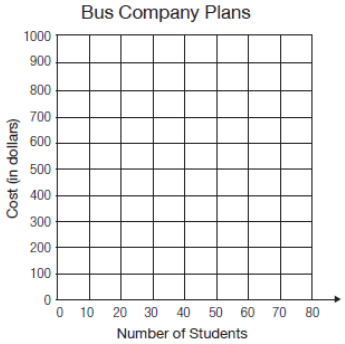
1. Which bus company plan do you think is the better deal? Explain why you think so.

2. Complete the table below.

Cost Comparison

Number of Students	Apple Charter	City Charter
5		
15		
25		
35		
45		

3. Now use the table to describe which bus company you think is the better deal, and compare it to your answer in question 1.



5. Describe in words how to find the cost of using Apple Charter Bus Company with any number of students:

The cost will equal _____ plus _____

_____

6. Use variables and numbers to write an equation that describes the cost of using Apple Charter Bus Company with any number of students. Let *C* represent the cost and *s* the number of students:

C = _____

_____. "Big Apple or Bust," *Student Math Notes*, National Council of Teachers of Mathematics. May/June 2009.

How Does Your Pattern Grow?

Student Explorations in Mathematics
Formerly Student Math Notes

Adding It Up

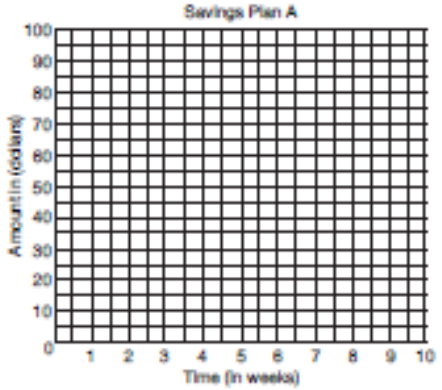
Imagine that you want to purchase the new *Larry Kotter and the Knights of Pythgoras* book on CD, which costs $49.95. You have only $25.50 right now. Your allowance is $5 per week.

1. First, guess how many weeks it will take you to save enough money for the CD.

2. Now use the table to determine how many weeks it will take you to save for the CD.

Week	Add	Total
0	–	$25.50
1	$5.00	$30.50
2		
3		
4		
5		
6		
7		
8		
9		

3. Using the table above and the grid in the next column, graph the information from both the number of weeks and the amount of money saved.



4. Should you connect the dots on your graph? Why, or why not?

5. What does point (8, 65.5) on the graph represent in this situation?

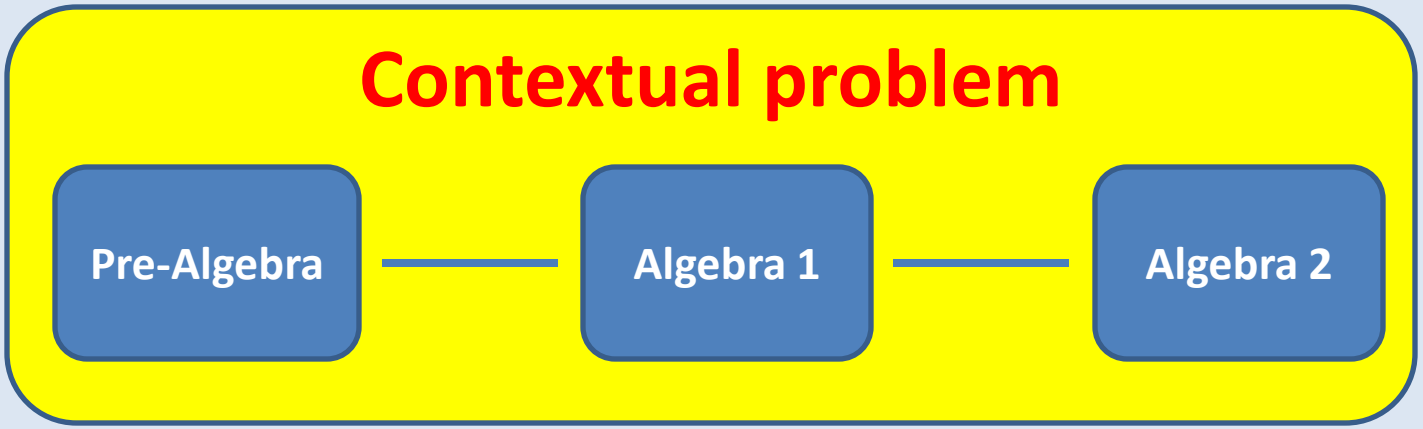
6. Use the table and the graph to determine how much money you will have at 10 weeks. Did you prefer using the table or the graph? Why?

This type of growth pattern in numbers—where you add a constant to each entry to obtain the next number—is called an **arithmetic sequence**. The pattern resembles a linear function but is not continuous.

_____. "How Does Your Pattern Grow?," *Student Explorations in Mathematics*, National Council of Teachers of Mathematics. September 2009.

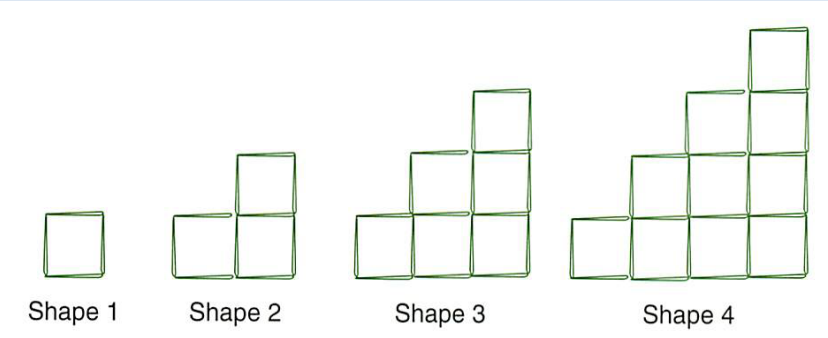
Problems with Access Points

We will look at some problems that have "multiple access points."
Teachers and students can enter at various points of the problem based on their background and readiness.



Pre-Algebra Access Point

Building with Toothpicks

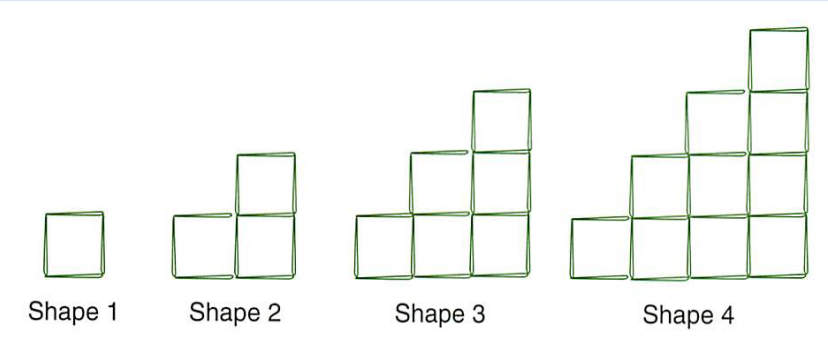


1. Use a pattern from the shapes to determine the perimeter of the fifth shape in the sequence. Show or explain how you arrived at your answer.
2. Write a formula that you could use to find the perimeter of any shape *n*. Explain how you found your formula.

Friel, Susan, Sid Rachlin, and Dot Doyle. Navigating through Algebra in grades 6-8. National Council of Teachers of Mathematics (NCTM): Reston, VA, 2001.

Algebra 1 Access Point

Building with Toothpicks



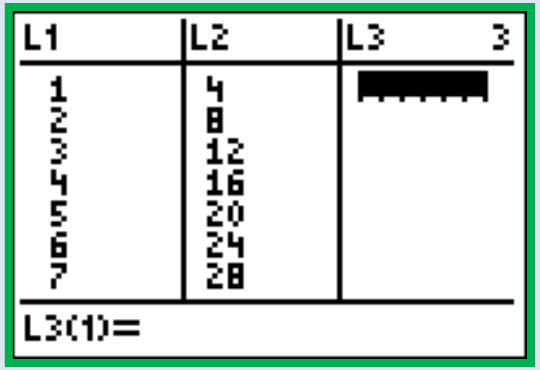
3. Create a table and a graph of the first seven shapes in the pattern. What rule did you use to continue the pattern? Explain how you determined your rule.

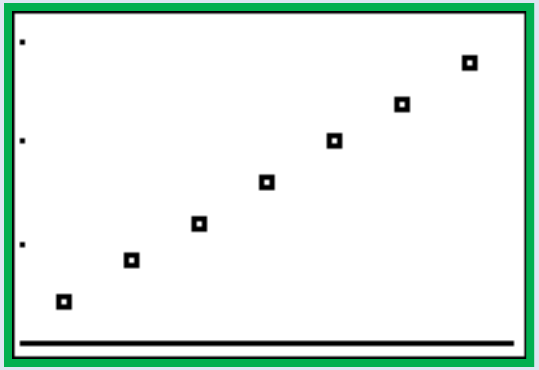
4. How would the pattern differ if you used triangles instead of squares?

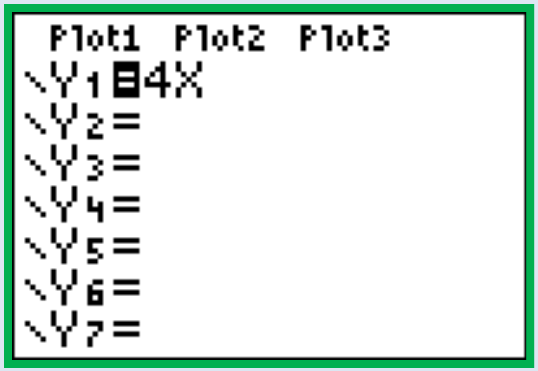
Extension from: Friel, Susan, Sid Rachlin, and Dot Doyle. Navigating through Algebra in grades 6-8. National Council of Teachers of Mathematics (NCTM): Reston, VA, 2001.

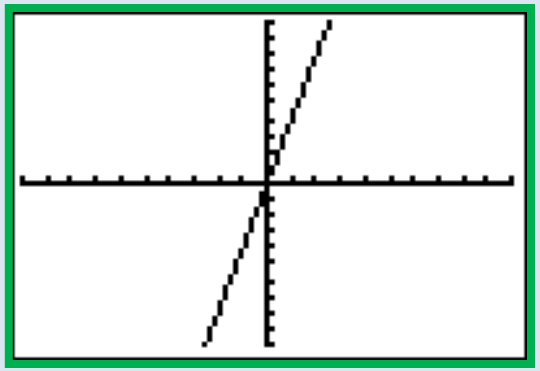
Algebra 1 Access Point

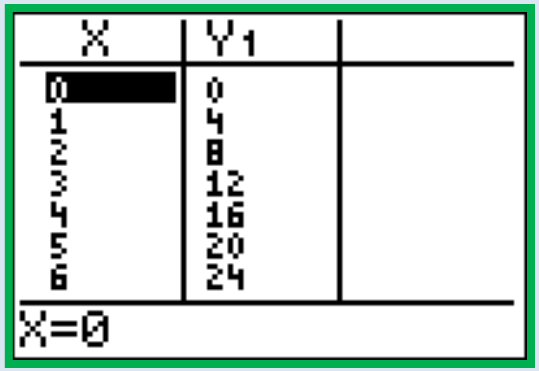
Building with Toothpicks





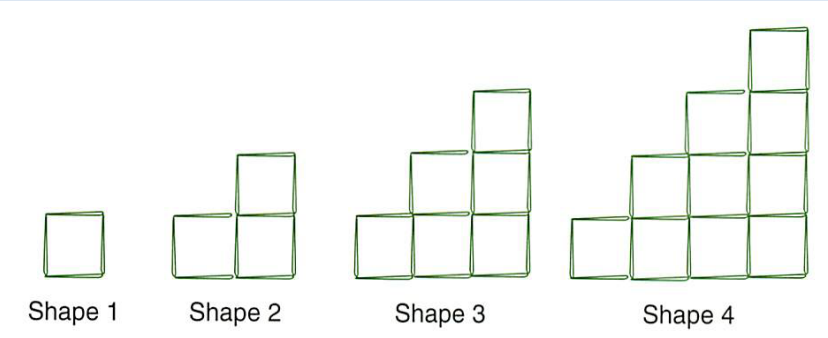






Algebra 2 Access Point

Building with Toothpicks



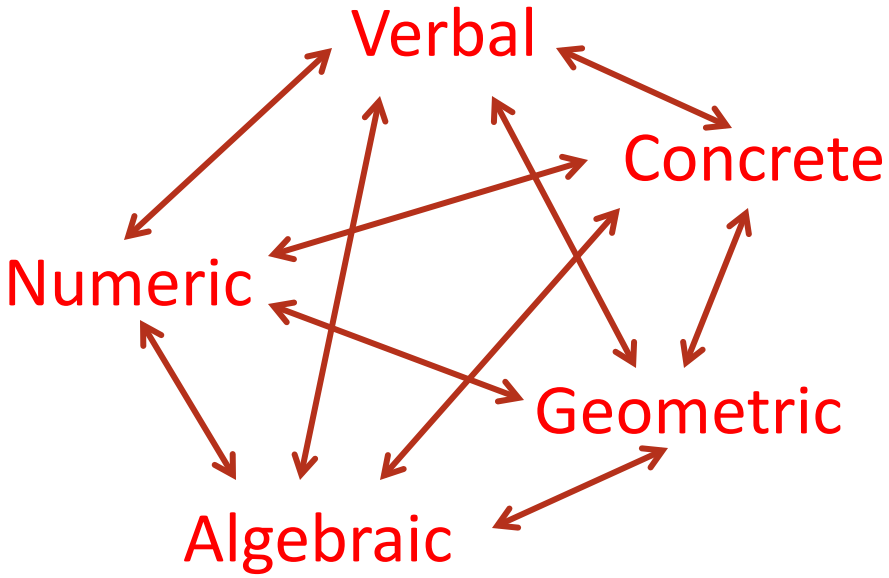
5. Determine the explicit and recursive formulas for finding the perimeter of the n^{th} figure.

6. What would be the perimeter of the 100^{th} figure?

Extension from: Friel, Susan, Sid Rachlin, and Dot Doyle. Navigating through Algebra in grades 6-8. National Council of Teachers of Mathematics (NCTM): Reston, VA, 2001.

Generalizing

We need to assist our students in moving among representations:



to generalize strategies and solve problems.

Reading and Questioning

The need to read problems carefully is vital – and to learn the importance of re-reading

Determine what is given in the problem and what is being asked

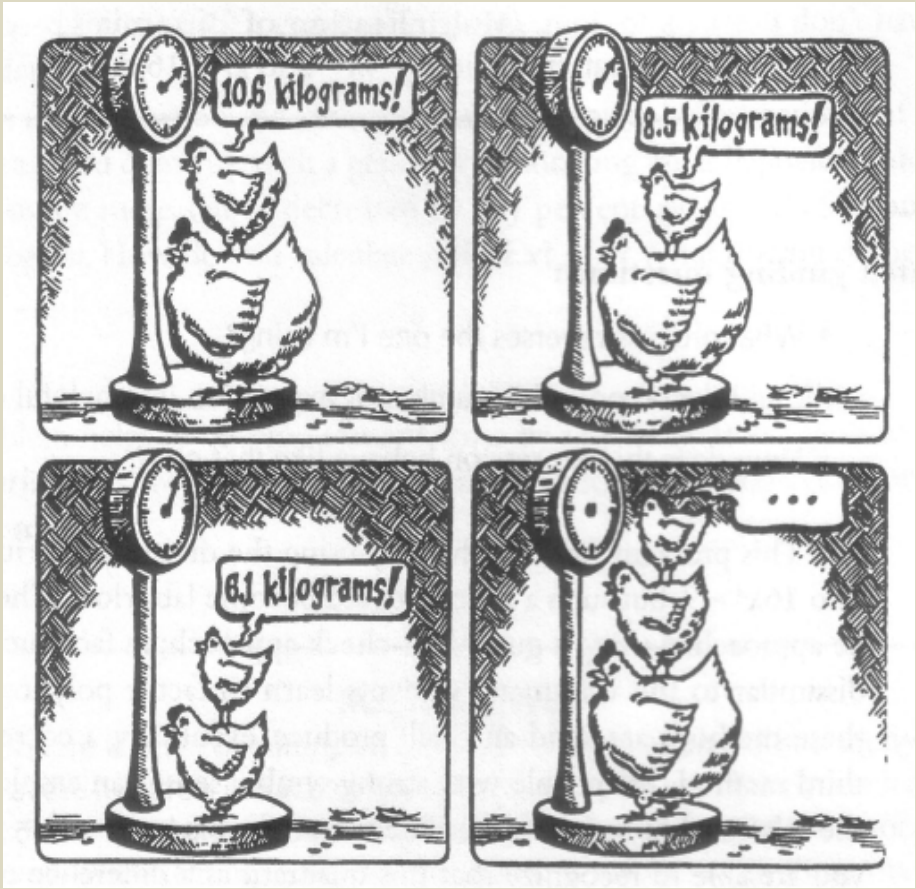
Get students to “unpack” their thinking and ask themselves questions

Ask probing questions to encourage student thinking

Encourage student reflection on their problem solving process

“Chicken/Scale” Problem

Three chickens were weighed in pairs; the first pair weighed in at 10.6 kg, the second pair weighed 8.5 kg, and the third pair weighed 6.1 kg. How much would the scale read if all three chickens were weighed at the same time? How many kilograms does each chicken weigh?



Driscoll, Mark. Fostering Algebraic Thinking, A Guide for Teachers Grades 6-10. Heinemann Publishing: Portsmouth, Hew Hampshire, 1999.

“Chicken/Scale” Solution

How might a student answer this problem?

“Chicken/Scale” Solution

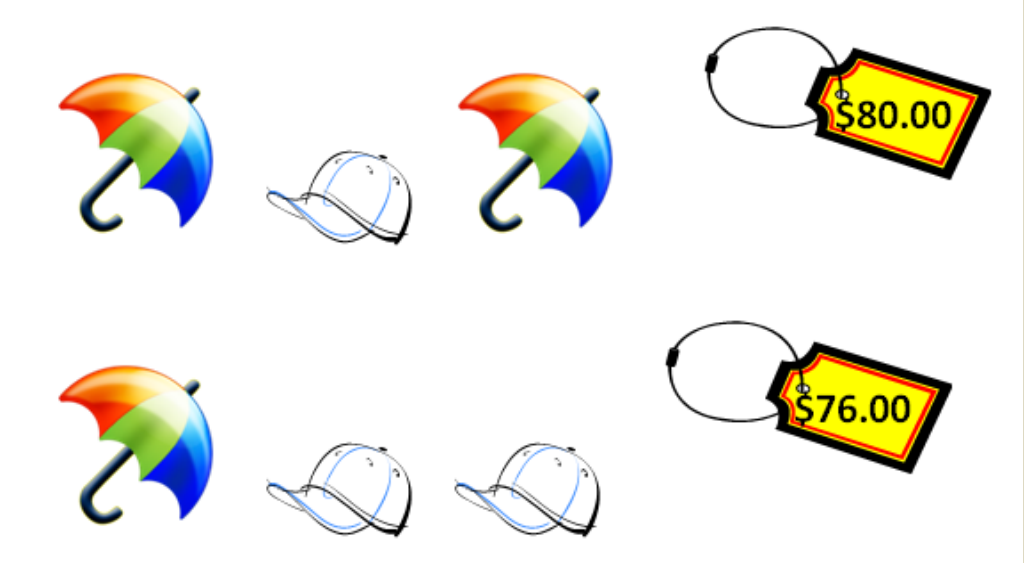
How might a student answer this problem?

Multiple pathways to solution

1. Guess and check
2. Patterns in tables of values
3. System of equations
4. Add three pictures together and divide by two

“Umbrella/Cap” Problem

Finding Prices. The following picture shows the costs of two combinations of umbrellas and hats:



What is the price of one umbrella? One cap?

Driscoll, Mark. Fostering Algebraic Thinking, A Guide for Teachers Grades 6-10. Heinemann Publishing: Portsmouth, Hew Hampshire, 1999.

“Umbrella/Cap” Problem

What other questions could be asked?

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“Umbrella/Cap” Problem

What other questions could be asked?

1. Without calculating the price of each, determine whether the cap or the umbrella is more expensive.
2. What is the difference in price between the cap and the umbrella?
3. Use the two pictures above to make a new combination of umbrellas and caps. Write down the cost of the combination.
4. Make a group of only caps and only umbrellas, then find its price.

Driscoll, Mark. Fostering Algebraic Thinking, A Guide for Teachers Grades 6-10. Heinemann Publishing: Portsmouth, Hew Hampshire, 1999.

"Umbrella/Cap" Problem

Possible umbrella and cap problem solutions

- Cover up a cap and an umbrella from each row and the difference in price between cap and umbrella is $4.00
- See two rows as pattern and add third row with three caps at $72. Divide to find 1 cap.
- Same pattern but add row above with three umbrellas at $84. Divide to find 1 umbrella.
- Create system of equations, 2u + 1c = 80 and u + 2c = 76.
- Might add them together to get 3u + 3c = 156.
  Divide by 3 to get u + c = 52

Friedlander, Alex and Michal Tabach, "Promoting Multiple Representations in Algebra" in Cuoco, Albert A. and Frances R. Curcio, editors. The Roles of Representation in School Mathematics. National Council of Teachers of Mathematics (NCTM): Reston, VA, 2001.

“Umbrella/Cap” Problem

Combination chart solution:

Follow pattern, then solve

Costs of Combinations (in dollars)

Number of Umbrellas							
	5						
	4						
	3	*84*					
	2		80				
	1			76			
	0				*72*		
		0	1	2	3	4	5
		Number of Caps					

Cuoco, Albert A. and Frances R. Curcio, editors. The Roles of Representation in School Mathematics. National Council of Teachers of Mathematics (NCTM): Reston, VA, 2001.

Patterns of Dots

A pattern of dots is shown below. At each step, more dots are added to the pattern. The number of dots added at each step is more than the number added in the previous step. The pattern continues indefinitely.

Step 1

Step 2

Step 3

How do you determine the number of dots in Step 20, but not have to draw all 20 pictures and then count the dots?

Explain or show how you could do this and give the answer that you get for the number of dots.

Problem adapted from Marcy's Dots problem, NAEP 1992.

Solutions?

- What did you do to solve the problem?
- What solution did you get? How can you justify your solution?
- What about the problem allows multiple solutions?

Solutions?

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- What solution did you get? How can you justify your solution?
- What about the problem allows multiple solutions?

Now, let's examine some student work

Examining Student Work

Sample 1

Sample 2

Sample 3

Sample 4

Sample 5

Sample 6

Solution 1

Patterns of Dots

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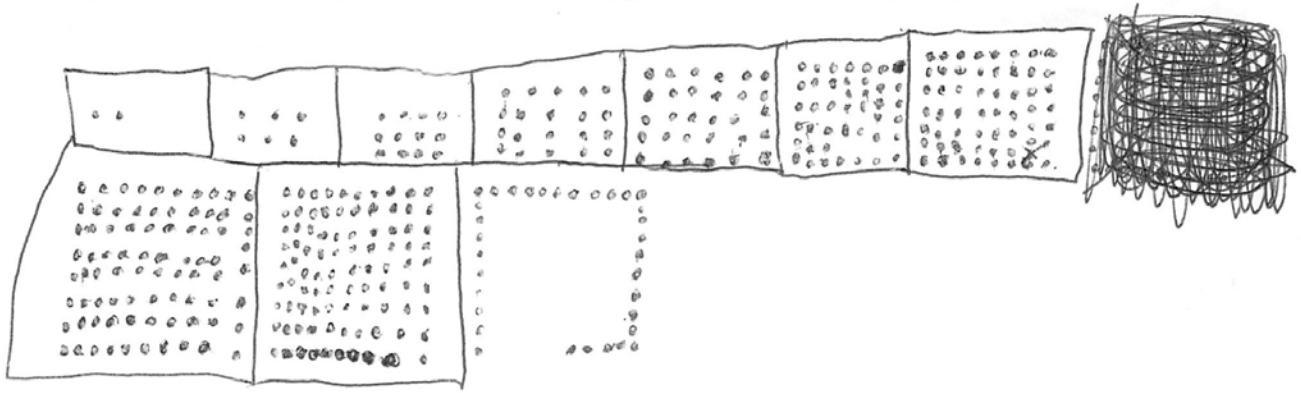
Step 1 Step 2 Step 3

How do you determine the number of dots in Step 20, but not have to draw all 20 pictures and then count the dots?

Explain or show how you could do this and give the answer that you get for the number of dots.



Implications of Solution Type 1

Student started a geometric progression

Next steps could include:

- Asking “What pattern do you notice?”
- Asking “Is there a pattern here? Can you think of a pattern that you could find so that you do not need to draw every step?”
- Move from geometric representation to numeric representation (symbolic).

Sample Menu

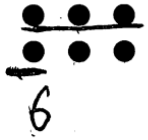
Solution 2

Patterns of Dots

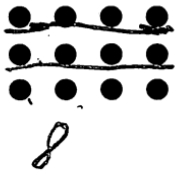
A pattern of dots is shown below. At each step, more dots are added to the pattern. The number of dots added at each step is more than the number added in the previous step. The pattern continues indefinitely.

Step 1

Step 2



Step 3



How do you determine the number of dots in Step 20, but not have to draw all 20 pictures and then count the dots?

Explain or show how you could do this and give the answer that you get for the number of dots.

4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42

I think 42 will be the number of dots.

Implications of Solution Type 2

Student found differences but ignored foundational number

Next steps could include:

- Asking “Why did you record 4, 6, 8?”
- Asking “What is the number of dots in steps 1, 2, and 3?”
- Asking if the answers are reasonable.
- Connecting numeric representation to geometric representation.

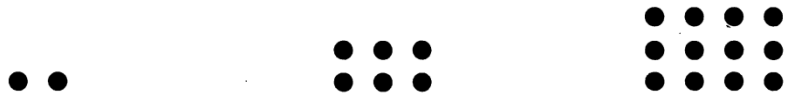
Sample Menu

Solution 3

Patterns of Dots

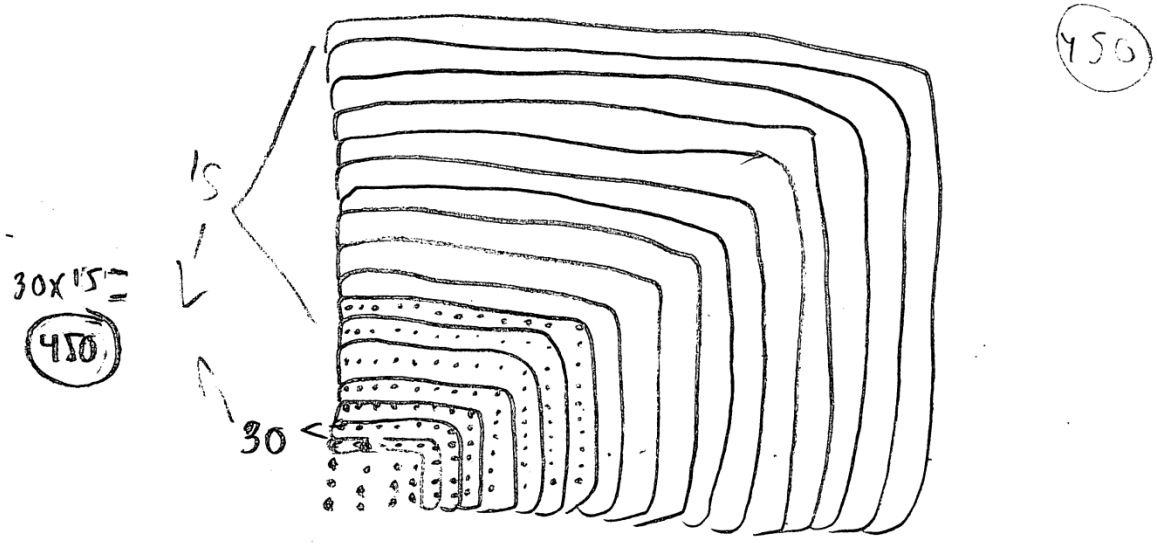
A pattern of dots is shown below. At each step, more dots are added to the pattern. The number of dots added at each step is more than the number added in the previous step. The pattern continues indefinitely.

Step 1 Step 2 Step 3



How do you determine the number of dots in Step 20, but not have to draw all 20 pictures and then count the dots?

Explain or show how you could do this and give the answer that you get for the number of dots.



Implications of Solution Type 3

Student identifies a geometric pattern

Next steps could include:

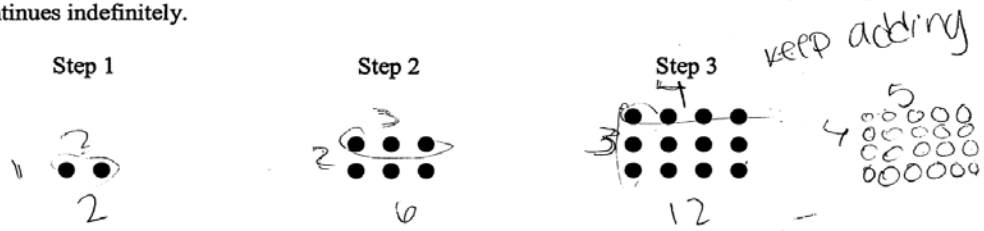
- Asking “How did you get 15 and 30?”
- Asking How does 15 and 30 fit with steps 1, 2, and 3?”
- Move from geometric representation to numeric representation (symbolic).

Sample Menu

Solution 4

A pattern of dots is shown below. At each step, more dots are added to the pattern. The number of dots added at each step is more than the number added in the previous step. The pattern continues indefinitely.

Step 1 | Step 2 | Step 3



How do you determine the number of dots in Step 20, but not have to draw all 20 pictures and then count the dots?

Explain or show how you could do this and give the answer that you get for the number of dots.

5 | 5,6 | 6,7 | 7,8 | 8,9 | 9,10 | 10,11 | 11,12 | 12,13 | 13,14 | 14,15 | 15,16

16 | 16,17 | 17,18 | 18,19 | 19,20 | 20,21

Implications of Solution Type 4

Student demonstrates geometric and numeric understanding

Next steps could include:

- Asking “What is your actual answer?”
- Asking “Did you check your answer?”
- Move to the next problem

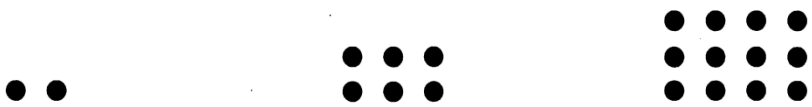
Sample Menu

Solution 5

Patterns of Dots

A pattern of dots is shown below. At each step, more dots are added to the pattern. The number of dots added at each step is more than the number added in the previous step. The pattern continues indefinitely.

Step 1 Step 2 Step 3



How do you determine the number of dots in Step 20, but not have to draw all 20 pictures and then count the dots?

Explain or show how you could do this and give the answer that you get for the number of dots.

2 6 12 16 32 36 72 76 152 156 312 316

+4 x2 +4 x2 +4 x2 +4 x2 +4 x2 +4 +2

660

Implications of Solution Type 5

Student found a numeric pattern, which is a repeating pattern instead of a building one, and does not align to the dot progression.

Next steps could include:

- Asking student to show the dot pattern for step 4.
- Asking "Did you determine the 20th step?"
- Connecting numeric representation to geometric representation.

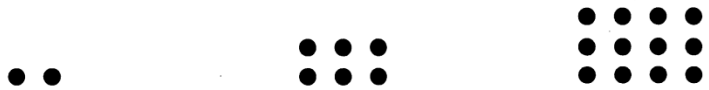
Sample Menu

Solution 6

Patterns of Dots

A pattern of dots is shown below. At each step, more dots are added to the pattern. The number of dots added at each step is more than the number added in the previous step. The pattern continues indefinitely.

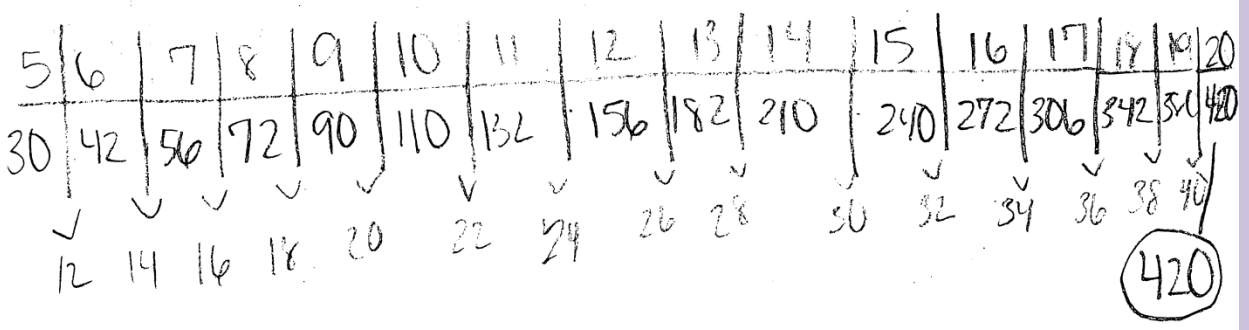
Step 1 Step 2 Step 3



How do you determine the number of dots in Step 20, but not have to draw all 20 pictures and then count the dots?

Explain or show how you could do this and give the answer that you get for the number of dots.

Step 1 has 1 row, step 2 has 2 rows, Step 3 has 3 rows, step 4 has 4 rows and so on. which would mean step 20 would have 20 rows.



Implications of Solution Type 6

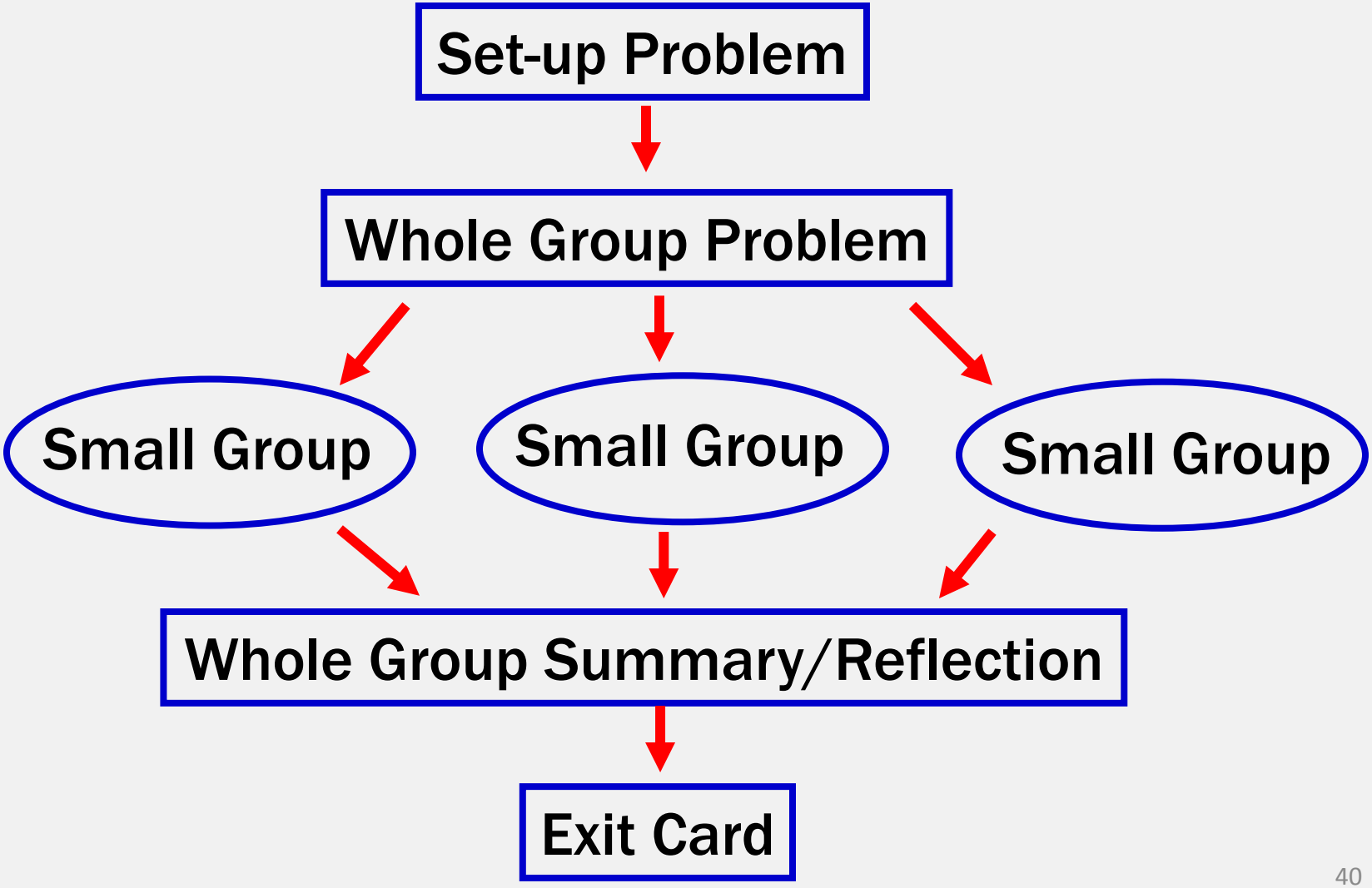
Student demonstrates understanding of differences.

Next steps could include:

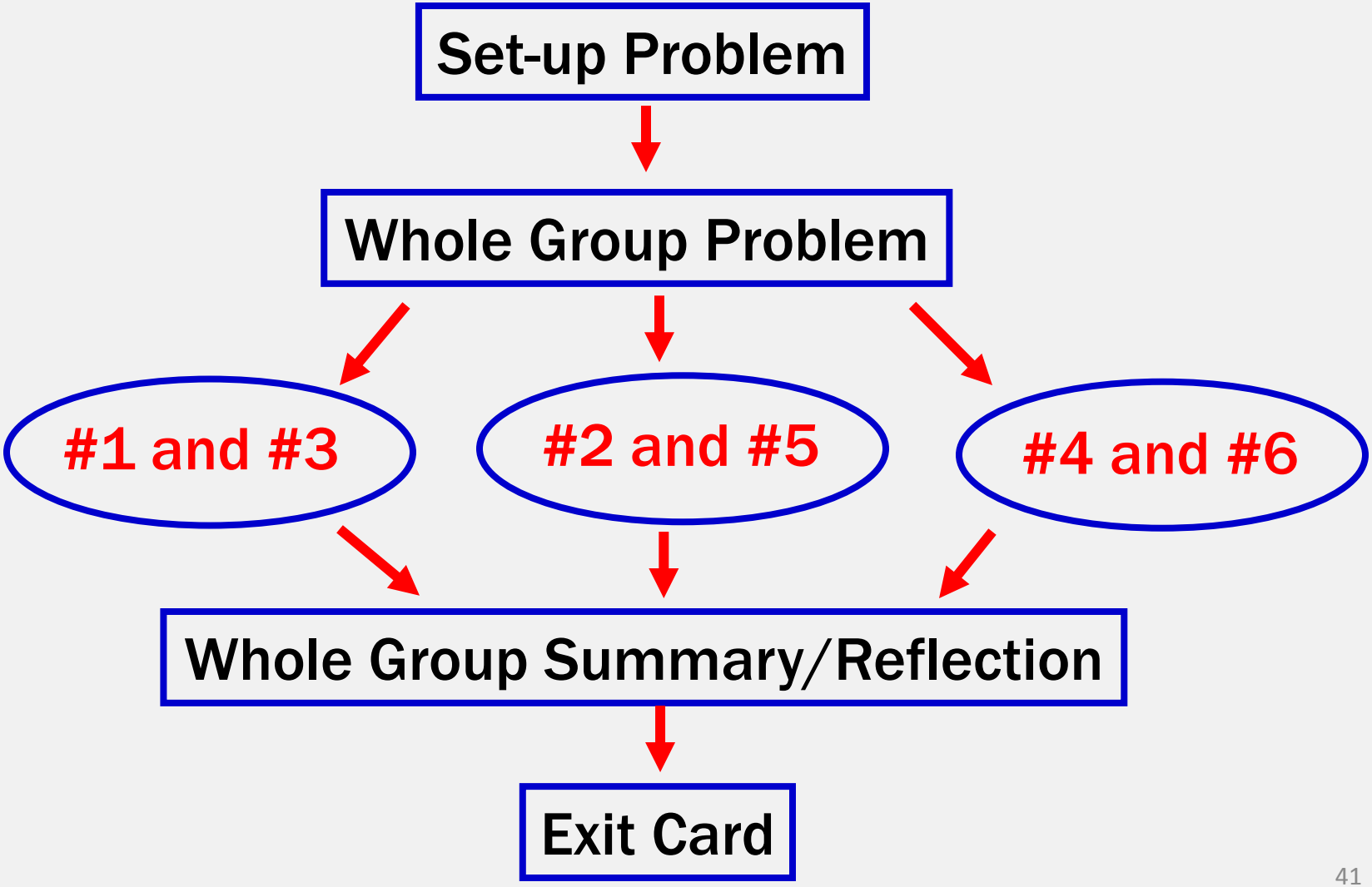
- Asking “What would the pattern of dots look like for step 20?”
- Asking “Could you get this answer without determining each step? What about determining the number of dots in step 50?”
- Move to the next problem.

Sample Menu

One Possible Lesson Design



One Possible Lesson Design



Summary

- What are you walking away with?

Goals:

- Look at engaging student activities
- Offer multiple access points for students
- How to use a student's representation
- Interpret their reasoning skills
- Determine the type of instruction that will be best suited for that particular type of learner

Targeting Student Reasoning and Sense Making in Algebra 1

Edward C. Nolan

ed@nolanmath.com

Montgomery County Public Schools

Rockville, Maryland